

A Construction of Compactly Supported Biorthogonal Scaling Vectors and Multiwavelets on \mathbf{R}^2

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Communicated by William A. Light

Received April 24, 2000; accepted in revised form April 1, 2002

In Kessler (*Appl. Comput. Harmonic Anal.* **9** (2000), 146–165), a construction was given for a class of orthogonal compactly supported scaling vectors on \mathbf{R}^2 , called short scaling vectors, and their associated multiwavelets. The span of the translates of the scaling functions along a triangular lattice includes continuous piecewise linear functions on the lattice, although the scaling functions are fractal interpolation functions and possibly nondifferentiable. In this paper, a similar construction will be used to create biorthogonal scaling vectors and their associated multiwavelets. The additional freedom will allow for one of the dual spaces to consist entirely of the continuous piecewise linear functions on a uniform subdivision of the original triangular lattice. © 2002 Elsevier Science (USA)

1. INTRODUCTION

Much research has been done in the construction of orthogonal multiresolution analyses of $L^2(\mathbf{R})$ (see [5, 10, 11]) and the associated multiwavelets (see [5, 8, 18]). All of the multiwavelet constructions have involved the completion of a matrix satisfying certain conditions. A class of nonseparable, orthogonal dilation-3 scaling functions defined on uniform triangulations of \mathbf{R}^2 were constructed in [6, 9]. (See [2, 13] for nonrelated constructions on \mathbf{R}^2 .) Multiwavelets for this specific example were found in [7]. In [15], the author generalized the construction of the orthogonal scaling vectors and provided a construction of the associated multiwavelets. This paper generalizes those results to construct a class of biorthogonal scaling vectors and the associated multiwavelets. The construction of the multiwavelets is analogous to the construction of multiwavelets for short scaling vectors introduced in [15].

1.1. Orthogonal Scaling Vectors and Multiwavelets

Let ε_1 and ε_2 be linearly independent vectors in \mathbf{R}^2 and let $\varepsilon_0 := (0, 0)$. For each $x \in \mathbf{R}^2$, there exist constants a and b such that $x = a\varepsilon_1 + b\varepsilon_2$. Then

define the metric $|x|_\varepsilon$ by

$$|x|_\varepsilon = \begin{cases} |a + b| & \text{if both } a, b \text{ have the same sign,} \\ \max\{|a|, |b|\} & \text{otherwise.} \end{cases}$$

Let \mathcal{T} be the 3-directional mesh with directions ε_1 , ε_2 , and $\varepsilon_2 - \varepsilon_1$. Define $\Delta_0 \in \mathcal{T}$ to be the triangular region with vertices ε_0 , ε_1 , and ε_2 , and $\nabla_0 \in \mathcal{T}$ to be the triangular region with vertices ε_1 , ε_2 , and $\varepsilon_1 + \varepsilon_2$. Define the translation function $t_{i,j}(x) := x - i\varepsilon_1 - j\varepsilon_2$ and the dilation function $d_{i,j}(x) := Nx - i\varepsilon_1 - j\varepsilon_2$ for some fixed integer dilation $N > 1$. Define the affine reflection function $r: \nabla_0 \rightarrow \Delta_0$ that maps the vertices ε_1 , ε_2 , and $\varepsilon_1 + \varepsilon_2$ to vertices ε_2 , ε_0 , and ε_1 , respectively. The notation $\check{f} := f \circ r$ is used for any f supported in Δ_0 .

DEFINITION. A *multiresolution analysis* (MRA) of $L^2(\mathbf{R}^2)$ of multiplicity r is a set of closed linear subspaces such that

1. $\dots \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \dots$
2. $\bigcap_{n \in \mathbf{Z}} V_n = \{0\}$.
3. $\overline{\bigcup_{n \in \mathbf{Z}} V_n} = L^2(\mathbf{R}^2)$.
4. $f \in V_n \Leftrightarrow f(N^{-n}\cdot) \in V_0$, $n \in \mathbf{Z}$.
5. There exists a set of functions $\{\phi^1, \phi^2, \dots, \phi^r\}$ such that $\{\phi^k \circ t_i : k = 1, \dots, r, i \in \mathbf{Z}^2\}$ forms a Riesz basis of V_0 .

The r -vector $\Phi := (\phi^1, \phi^2, \dots, \phi^r)^T$ is referred to as a *scaling vector* and the individual ϕ^k as *scaling functions*.

Conditions 1, 4, and 5 imply that a scaling vector Φ with compactly supported ϕ^k satisfies the dilation equation

$$\Phi(x) = N \sum_{i \in \mathbf{Z}^2} g_i \Phi \circ d_i \tag{1.1}$$

for a finite number of $r \times r$ scalar matrices g_i .

DEFINITION. A vector Φ of r linearly independent functions on \mathbf{R}^2 is *refinable* at dilation N if it satisfies (1.1) for some sequence of $r \times r$ scalar matrices c_i .

A simple example of a MRA of $L^2(\mathbf{R}^2)$ over the mesh \mathcal{T} is constructed by defining the ‘‘hat’’ function h by

$$h(x) := \begin{cases} 1 - |x|_\varepsilon & \text{for } |x|_\varepsilon < 1, \\ 0 & \text{otherwise,} \end{cases} \tag{1.2}$$

and letting $\Phi = \{h\}$. Using the notation

$$S(H) := \text{clos}_{L^2} \text{span} \{f \circ t_i : i \in \mathbf{Z}^2, f \in H\} \quad \text{for } H \subset L^2(\mathbf{R}^2),$$

then define $V_0 := S(\Phi)$. It is easily verified that the scaling vector is refinable for any integer dilation $N > 1$, and that (V_p) is a MRA, where $V_p := S(\Phi(N^p \cdot))$.

For function vectors Γ and Λ with elements in $L^2(\mathbf{R}^2)$, define

$$\langle \Gamma, \Lambda \rangle = \int_{\mathbf{R}^2} \Gamma(x)\Lambda(x)^T dx.$$

DEFINITION. If $\langle \Phi, \Phi \circ t_{i,j} \rangle = \delta_{0,i}\delta_{0,j}I$, then we say that Φ is an *orthogonal scaling vector*. If the ϕ^k are compactly supported, then the multiresolution analysis generated by Φ is said to be *orthogonal*.

Define W_n to be the orthogonal complement of V_n in V_{n+1} , so that

$$V_{n+1} = V_n \oplus W_n \quad \text{for } n \in \mathbf{Z}.$$

The W_n , referred to as wavelet spaces, are necessarily pairwise orthogonal and are spanned by the orthogonal dilations and translations of a set of functions $\{\psi^1, \psi^2, \dots, \psi^t\}$, referred to as wavelets, that satisfy the equation

$$\Psi(x) = N \sum_{i \in \mathbf{Z}^2} h_i \Phi \circ d_i \tag{1.3}$$

for some h_i , where Ψ is the t -vector $(\psi^1, \psi^2, \dots, \psi^t)^T$.

DEFINITION. A pair of n -dimensional function vectors Φ and $\tilde{\Phi}$ are said to be *biorthogonal* if

$$\langle \Phi, \tilde{\Phi} \circ t_{i,j} \rangle = \delta_{0,i}\delta_{0,j}I, \quad i, j \in \mathbf{Z}.$$

A necessary and sufficient condition for the construction of biorthogonal vectors was given in [12], and will be stated here without proof.

LEMMA 1.1 *Suppose U and W are m -dimensional subspaces of \mathbf{R}^n . There exist dual (biorthogonal) bases for U and W if and only if $U \cap W^\perp = \{0\}$.*

If the criteria of Lemma 1.1 are met, then the Gram-Schmidt orthogonalization process can be modified to extract biorthogonal sets from bases for U and W .

1.2. Short Scaling Vectors

Throughout the paper, P_K denotes the orthogonal projection onto a subspace K of $L^2(\mathbf{R}^2)$. If C is a compact set of \mathbf{R}^2 and U is a space of functions on \mathbf{R}^2 , then define

$$U(C) := \{f \in U : \text{supp}(f) \subseteq C\}.$$

DEFINITION. Suppose $\bar{\Phi} = (\phi^1, \dots, \phi^s)^T$ is refinable. If $w = (w^1, \dots, w^t)^T$ is such that $\Phi = (w^1, \dots, w^t, \phi^1, \dots, \phi^s)^T$ is refinable, then w is said to extend $\bar{\Phi}$.

The construction of short scaling vectors is given in [15], and is a generalization of a construction in [6]. Recall the nonorthogonal scaling vector generated by the ‘‘hat’’ function h defined in (1.2). Define $h_i := h(x - \varepsilon_i)|_{\Delta_0}$, $i = 0, 1, 2$.

DEFINITION. Suppose there is a subspace \mathcal{W} of $C_0(\mathbf{R}^2)$ with an orthogonal basis $\{w^1, \dots, w^t\}$ such that

1. $B := \{w^1, \dots, w^t, \check{w}^1, \dots, \check{w}^t\}$ extends $\{h\}$,
2. $\text{supp}(w^i) \subseteq \Delta_0$, $i = 1, \dots, t$,
3. $(I - P_{\mathcal{W}})h_i \perp (I - P_{\mathcal{W}})h_j$, $i \neq j$, $i, j \in \{0, 1, 2\}$.

Then

$$\Phi := \{w^1, \dots, w^t, \check{w}^1, \dots, \check{w}^t, (I - P_{S(B)})h\}^T$$

is called a *short* scaling vector, and generates a MRA (V_p) of multiplicity $q := 2t + 1$ such that V_0 still includes continuous piecewise linear functions on \mathcal{T} .

A dilation-3 example with $\mathcal{W} = \{w\}$ is given in [15] and is illustrated in Fig. 1. The scaling functions are fractal interpolation surfaces that are nondifferentiable and have a nonintegral box dimension greater than 2. (A full introduction to fractal interpolation surfaces can be found in [9, 17].)

2. MAIN RESULTS

Suppose that X and Y are spaces spanned by biorthogonal function vectors. Then define the projection operator P_X^Y such that $\ker P_X^Y = Y^\perp$ and $\text{range } P_X^Y = X$. If $\mathcal{X} := S(X)$ and $\mathcal{Y} := S(Y)$ are finite shift invariant

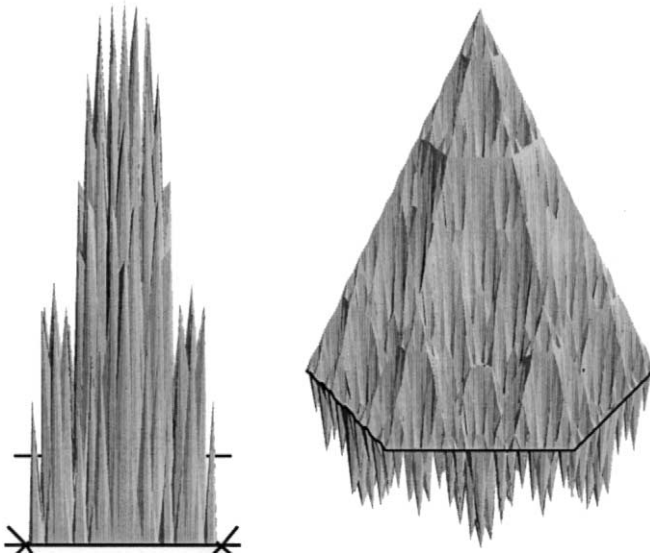


FIG. 1. Approximations to scaling functions from a dilation-3 short scaling vector.

spaces, then

$$P_x^{y^j} f := \sum_{j \in \mathbf{Z}^2} \sum_{i=1}^n \frac{\langle f, y^i \circ t_j \rangle}{\langle x^i, y^i \rangle} x^i \circ t_j,$$

where $x_i \in X$ and $y_i \in Y$.

Define $h, h_0, h_1,$ and h_2 as in Section 1.2. Then we have the following result.

THEOREM 2.1. *Suppose there are function vectors $B := \{w^1, \dots, w^t\}$ and $\tilde{B} := \{\tilde{w}^1, \dots, \tilde{w}^t\}$ in $C_0(\mathbf{R}^2)$ such that*

1. B and \tilde{B} are biorthogonal,
2. $\{w^1, \dots, w^t, \check{w}^1, \dots, \check{w}^t\}$ and $\{\tilde{w}^1, \dots, \tilde{w}^t, \check{\tilde{w}}^1, \dots, \check{\tilde{w}}^t\}$ each extend $\{h\}$,
3. $\text{supp}(w^i), \text{supp}(\tilde{w}^i) \subseteq \Delta_0, i = 1, \dots, t,$ and
4. $(I - P_{\mathcal{W}}^{y^i})h_i \perp (I - P_{\check{\mathcal{W}}}^{y^j})h_j, i \neq j, i, j \in \{0, 1, 2\},$ where $\mathcal{W} = S(B)$ and $\check{\mathcal{W}} = S(\tilde{B})$.

Then there exist biorthogonal scaling vectors Φ and $\check{\Phi}$ of length $q := 2t + 1$ such that $V_0 := S(\Phi)$ and $\check{V}_0 := S(\check{\Phi})$ each contain continuous piecewise linear functions on the mesh \mathcal{T} .

Proof. The main issue is finding compactly supported functions ϕ^i and $\tilde{\phi}^j$ that satisfy the biorthogonality conditions $\langle \phi^i, \tilde{\phi}^j \rangle = \delta_{i,j}$. Define the following:

$$\phi^i := w^i \quad \text{for } i = 1, \dots, t, \quad \tilde{\phi}^i := \tilde{w}^i \quad \text{for } i = 1, \dots, t,$$

$$\phi^{t+i} := \check{w}^i \quad \text{for } i = 1, \dots, t \quad \text{and} \quad \tilde{\phi}^{t+i} := \check{\tilde{w}}^i \quad \text{for } i = 1, \dots, t,$$

$$\phi^q := \frac{1}{\alpha} (I - P_{\mathcal{W}'}^{\mathcal{W}'})h, \quad \tilde{\phi}^q := \frac{1}{\beta} (I - P_{\check{\mathcal{W}'}}^{\check{\mathcal{W}'}})h,$$

where α, β are constants such that $\alpha\beta := \langle (I - P_{\mathcal{W}'}^{\mathcal{W}'})h, (I - P_{\check{\mathcal{W}'}}^{\check{\mathcal{W}'}})h \rangle$. Let $\Phi := (\phi^1, \dots, \phi^q)^T$ and $\tilde{\Phi} := (\tilde{\phi}^1, \dots, \tilde{\phi}^q)^T$. Then set $V_p := S(\Phi(N^{P \cdot}))$ and $\tilde{V}_p := S(\tilde{\Phi}(N^{P \cdot}))$.

Condition (1) guarantees that

$$\begin{aligned} \langle \phi^i, \tilde{\phi}^j \rangle &= \delta_{i,j} \quad \text{for } i, j = 1, \dots, t \quad \text{and} \\ \langle \phi^i, \tilde{\phi}^j \rangle &= \delta_{i,j} \quad \text{for } i, j = t + 1, \dots, 2t. \end{aligned}$$

Condition (3) guarantees that

$$\begin{aligned} \langle \phi^i, \tilde{\phi}^j \rangle &= 0 \quad \text{for } i = 1, \dots, t, \quad j = t + 1, \dots, 2t \quad \text{and} \\ \langle \phi^i, \tilde{\phi}^j \rangle &= 0 \quad \text{for } i = t + 1, \dots, 2t, \quad j = 1, \dots, t. \end{aligned}$$

Condition (4) establishes the remaining orthogonality conditions:

$$\begin{aligned} \langle \phi^q, \tilde{\phi}^i \rangle &= 0 \quad \text{for } i = 1, \dots, 2t \quad \text{and} \\ \langle \phi^i, \tilde{\phi}^q \rangle &= 0 \quad \text{for } i = 1, \dots, 2t. \end{aligned}$$

Condition (2) guarantees that both Φ and $\tilde{\Phi}$ are refinable, and that $V_n \subset V_{n+1}$ and $\tilde{V}_n \subset \tilde{V}_{n+1}$. The requirements that $\bigcap_{j \in \mathbb{Z}} V_j = 0$, $\bigcap_{j \in \mathbb{Z}} \tilde{V}_j = 0$, $\overline{\bigcup_{j \in \mathbb{Z}} V_j} = L^2(\mathbf{R})$, and $\overline{\bigcup_{j \in \mathbb{Z}} \tilde{V}_j} = L^2(\mathbf{R})$, and that the translates of Φ and $\tilde{\Phi}$ form Riesz bases, are trivially met by compactly supported scaling vectors. Therefore, both (V_p) and (\tilde{V}_p) are MRAs. ■

While the restrictions on the spaces B and \tilde{B} are extensive, such spaces do exist. An example is provided in Section 3. It is important to note that, as with the short scaling vectors, the scaling functions and associated multiwavelets in these constructions may be nondifferentiable FIS. However, by relaxing the need for orthogonality, it is possible to construct

a “smoother” basis that can be used in the reconstruction phase of applications.

Section 4 will give a detailed definition of the wavelet spaces $W_f, \tilde{W}_f, W_g, \tilde{W}_g, W_h,$ and \tilde{W}_h . W_f and \tilde{W}_f will have generators supported on triangles, W_g and \tilde{W}_g will have generators supported on parallelograms, and W_h and \tilde{W}_h will have generators supported on hexagons. The main theorem on the construction of the $q(N^2 - 1)$ wavelets (see [14]) will be stated and proven in that section.

3. A CONSTRUCTION OF BIORTHOGONAL SCALING VECTORS

Set vectors $\varepsilon_1 := (1, 0)$ and $\varepsilon_2 := (1/2, \sqrt{3}/2)$, so that \mathcal{T} is a 3-directional mesh of equilateral triangles. Let h be the generalized hat function defined in (1.2) and fix $N = 3$. In order to construct a scaling vector Φ that satisfies Theorem 2.1, let w and u be continuous functions with (nonempty) support in Δ_0 and let $\check{w} := w \circ r$ and $\check{u} := u \circ r$. Let $G = \{(0, 0), (1, 0), (2, 0), (0, 1), (1, 1), (0, 2)\}$ and let $\check{G} = \{(0, 0), (1, 0), (0, 1)\}$. With condition 2 of Theorem 2.1 in mind, we require that w and u satisfy the following dilation equations for some $\alpha, \beta, s_i, \check{s}_i, q_i, \check{q}_i$:

$$w = \alpha h \circ d_{1,1} + \sum_{i \in G} s_i w \circ d_i + \sum_{i \in \check{G}} \check{s}_i \check{w} \circ d_i, \tag{3.1}$$

$$u = \beta h \circ d_{1,1} + \sum_{i \in G} q_i u \circ d_i + \sum_{i \in \check{G}} \check{q}_i \check{u} \circ d_i. \tag{3.2}$$

The functions w and u are FIS, with interpolation points located uniformly over Δ_0 as illustrated in Fig. 2, provided $|s_i| < 1$ and $|q_i| < 1$ for all $i \in \mathbb{Z}^2$ (this is necessary if w and u are to be continuous).

In order to construct $w, \check{w}, u,$ and \check{u} with rotational symmetry about the centroid of their support triangle, let

$$\begin{aligned} s_{0,0} = s_{2,0} = s_{0,2} &:= s_1, & q_{0,0} = q_{2,0} = q_{0,2} &:= q_1, \\ s_{1,0} = s_{0,1} = s_{1,1} &:= s_2 \quad \text{and} & q_{1,0} = q_{0,1} = q_{1,1} &:= q_2, \\ \check{s}_{0,0} = \check{s}_{1,0} = \check{s}_{0,1} &:= s_3, & \check{q}_{0,0} = \check{q}_{1,0} = \check{q}_{0,1} &:= q_3, \end{aligned}$$

where $|s_i|, |q_i| < 1$ for $i = 1, 2, 3$. Then the only free parameters will be the scaling variables $s_i, q_i,$ and α and $\beta,$ the values of the functions w and u at the centroid of $\Delta_0,$ respectively. Set $\alpha, \beta := 1$ for this construction.

Recall that $h_i = h(\cdot - \varepsilon_i)|_{\Delta_0},$ where $i = 0, 1, 2.$ Due to the rotational invariance of both w and the set of h_i 's, the six orthogonality conditions

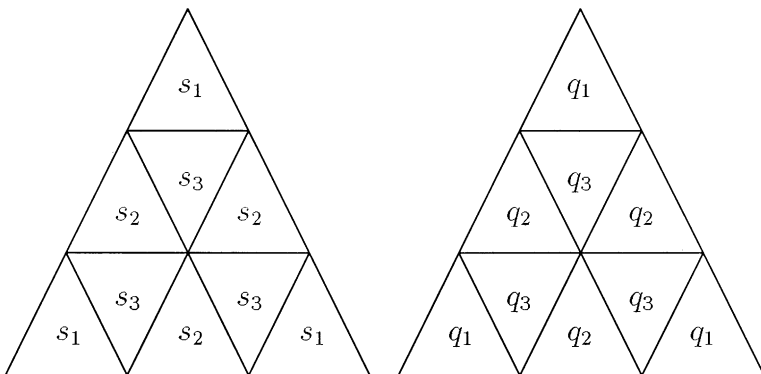


FIG. 2. The domain points and scalings used in the example.

needed to satisfy condition 4 of Theorem 2.1 reduce to just

$$(I - P_{\mathcal{W}}^{\tilde{\mathcal{W}}})h_0 \perp (I - P_{\tilde{\mathcal{W}}}^{\mathcal{W}})h_1, \tag{3.3}$$

where $\mathcal{W} = S(w)$ and $\tilde{\mathcal{W}} = S(u)$. Since

$$P_{\mathcal{W}}^{\tilde{\mathcal{W}}}h_0 = \frac{\langle h_0, u \rangle}{\langle w, u \rangle} w \quad \text{and} \quad P_{\tilde{\mathcal{W}}}^{\mathcal{W}}h_1 = \frac{\langle h_1, w \rangle}{\langle w, u \rangle} u,$$

then (3.3) reduces to

$$\langle h_0, h_1 \rangle = \frac{\langle h_0, u \rangle \langle h_1, w \rangle}{\langle w, u \rangle}. \tag{3.4}$$

Since $\langle h_0, w \rangle = \langle h_1, w \rangle = \langle h_2, w \rangle$ and $h_0 + h_1 + h_2 = 1$ on Δ_0 , we calculate $\langle w, 1 \rangle$ using (3.1):

$$\begin{aligned} \langle w, 1 \rangle &= \langle h \circ d_{1,1}, 1 \rangle + \sum_{i \in G} s_i \langle w \circ d_i, 1 \rangle + \sum_{i \in \tilde{G}} \check{s}_i \langle \check{w} \circ d_i, 1 \rangle \\ &= \frac{\langle h, 1 \rangle}{3(3 - \sum_{i=1}^3 s_i)}. \end{aligned}$$

Likewise, from (3.2)

$$\langle u, 1 \rangle = \frac{\langle h, 1 \rangle}{3(3 - \sum_{i=1}^3 q_i)}.$$

Since $\langle h, 1 \rangle = \frac{\sqrt{3}}{2}$, $\langle h_0, w \rangle = \frac{1}{3}\langle w, 1 \rangle$, and $\langle h_1, u \rangle = \frac{1}{3}\langle u, 1 \rangle$, then

$$\langle h_0, w \rangle = \frac{\sqrt{3}}{18(3 - \sum_{i=1}^3 s_i)} \quad \text{and} \quad \langle h_1, u \rangle = \frac{\sqrt{3}}{18(3 - \sum_{i=1}^3 q_i)}. \quad (3.5)$$

Again, using both (3.1) and (3.2),

$$\langle w, u \rangle = \frac{\langle h, h \rangle + 3(s_2 + s_3)\langle h_0, w \rangle + 3(q_2 + q_3)\langle h_0, u \rangle}{3(3 - \sum_{i=1}^3 s_i q_i)}.$$

Since $\langle h, h \rangle = \frac{\sqrt{3}}{4}$, then using (3.5),

$$\begin{aligned} \langle w, u \rangle &= \frac{\sqrt{3}[3(3 - \sum_{i=1}^3 s_i)(3 - \sum_{i=1}^3 q_i) + 2(s_2 + s_3)(3 - \sum_{i=1}^3 q_i) + 2(q_2 + q_3)(3 - \sum_{i=1}^3 s_i)]}{36(3 - \sum_{i=1}^3 s_i)(3 - \sum_{i=1}^3 q_i)(3 - \sum_{i=1}^3 s_i q_i)}. \end{aligned} \quad (3.6)$$

Substituting (3.5), (3.6), and $\langle h_0, h_1 \rangle = \frac{\sqrt{3}}{48}$ into (3.4) and requiring that (3.6) is nonzero provides the following necessary conditions on the s_i 's and q_i 's:

$$\begin{aligned} 27s_1 + 9s_2 + 9s_3 + 27q_1 + 9q_2 + 9q_3 - 25s_1q_1 - 3s_1q_2 - 3s_1q_3 - 3s_2q_1 \\ - 13s_2q_2 + 3s_2q_3 - 3s_3q_1 + 3s_3q_2 - 13s_3q_3 - 33 = 0, \quad \text{and} \end{aligned} \quad (3.7)$$

$$\begin{aligned} 27 - 9q_1 - 3q_2 - 3q_3 - 9s_1 + 3q_1s_1 + q_2s_1 + q_3s_1 - 3s_2 + q_1s_2 \\ - q_2s_2 - q_3s_2 - 3s_3 + q_1s_3 - q_2s_3 - q_3s_3 \neq 0. \end{aligned} \quad (3.8)$$

By letting $s_i := 0$ for $i = 1, 2, 3$, w becomes piecewise linear and (3.7) and (3.8) reduce to

$$3(9q_1 + 3q_2 + 3q_3 - 11) = 0 \quad \text{and} \quad 3(9 - 3q_1 - q_2 - q_3) \neq 0. \quad (3.9)$$

Furthermore, by letting $q =: q_i$ for $i = 1, 2, 3$, (3.9) reduces to $45q - 33 = 0$ and $27 - 15q \neq 0$, with the solution $q = 11/15$.

Define the scaling functions

$$\begin{aligned} \phi^1 &:= w, \\ \phi^2 &:= \check{w}, \\ \phi^3 &:= h - \sum_{i \in H} \frac{\langle h, u \circ t_i \rangle}{\langle w, u \rangle} w \circ t_i - \sum_{i \in \check{H}} \frac{\langle h, \check{u} \circ t_i \rangle}{\langle w, u \rangle} \check{w} \circ t_i, \\ \tilde{\phi}^1 &:= \frac{u}{\langle w, u \rangle}, \end{aligned}$$

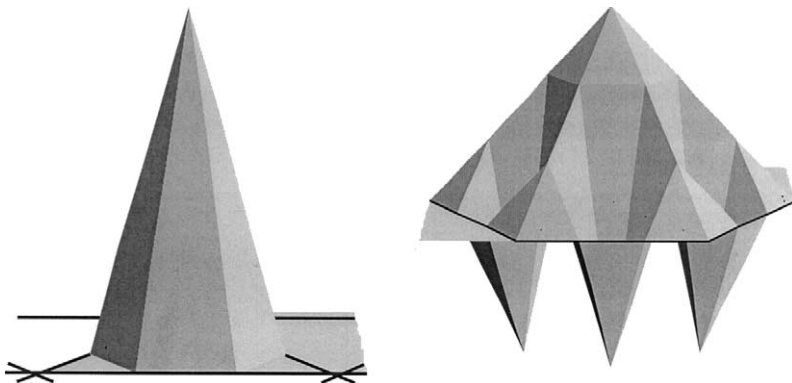


FIG. 3. Scaling functions ϕ^1 and ϕ^3 with $s_i = 0$.

$$\begin{aligned} \tilde{\phi}^2 &:= \frac{\check{u}}{\langle w, u \rangle}, \\ \tilde{\phi}^3 &:= \frac{1}{\alpha} \left[h - \sum_{i \in H} \frac{\langle h, w \circ t_i \rangle}{\langle w, u \rangle} u \circ t_i - \sum_{i \in \check{H}} \frac{\langle h, \check{w} \circ t_i \rangle}{\langle w, u \rangle} \check{u} \circ t_i \right], \end{aligned}$$

where $H = \{(0, 0), (0, -1), (-1, 0)\}$ and $\check{H} = \{(0, -1), (-1, 0), (-1, -1)\}$, and

$$\alpha := 6 \left(\langle h_0, h_0 \rangle - \frac{\langle h_0, w \rangle \langle h_0, u \rangle}{\langle w, u \rangle} \right).$$

Then $\Phi := (\phi^1, \phi^2, \phi^3)^T$ and $\tilde{\Phi} := (\tilde{\phi}^1, \tilde{\phi}^2, \tilde{\phi}^3)^T$ are biorthogonal scaling vectors that generate the MRAs $V_p = S(\Phi(3^p(x, y)))$ and $\tilde{V}_p = S(\tilde{\Phi}(3^p(x, y)))$. Note that both V_0 and \tilde{V}_0 contain piecewise linears on the triangulation \mathcal{T} and with $s_i = 0$ for $i = 1, 2, 3$, V_0 is the set of piecewise linears on a uniform subdivision of \mathcal{T} . This set of scaling functions and their biorthogonal counterparts with $q_i = 11/15$ for $i = 1, 2, 3$ are illustrated in Figs. 3 and 4.

Notice that for any nonsingular linear map A , we may define the same set of scaling functions on the lattice generated by Ae_1 and Ae_2 , and the functions will maintain their biorthogonality.

4. CONSTRUCTION OF ASSOCIATED MULTIWAVELETS

Let Φ and $\tilde{\Phi}$ be the scaling vectors constructed in Theorem 2.1 and let (V_p) and (\tilde{V}_p) be the corresponding MRAs. Recall that $\text{supp } \phi^i, \text{supp } \tilde{\phi}^i \subseteq \Delta_0 = \Delta(\varepsilon_0, \varepsilon_1, \varepsilon_2) \in \mathcal{T}$ for $i = 1, \dots, t$ and that $\text{supp } \phi^i, \text{supp } \tilde{\phi}^i \subseteq \nabla_0 = \Delta(\varepsilon_1, \varepsilon_2, \varepsilon_1 + \varepsilon_2) \in \mathcal{T}$ for $i = t + 1, \dots, 2t$. First consider wavelets supported in $\Delta \in \mathcal{T}$.

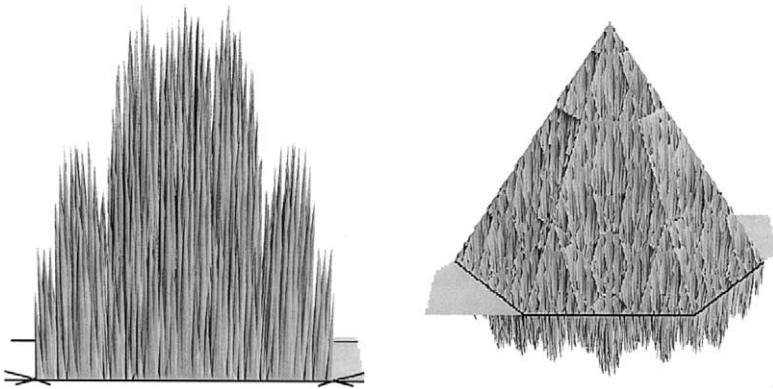


FIG. 4. Approximations to scaling functions $\tilde{\phi}^1$ and $\tilde{\phi}^3$ with $q_i = 11/15$.

Consider the $(tN^2 + \frac{(N-1)(N-2)}{2})$ -dimensional spaces $V_1(\Delta_0)$ and $\tilde{V}_1(\Delta_0)$, with the bases consisting of t dilated scaling functions on each of the N^2 subtriangles and $\frac{(N-1)(N-2)}{2}$ dilated ϕ^q and $\tilde{\phi}^q$.

Define the functions

$$g_i := P_{V_1(\Delta_0)}^{\tilde{V}_1(\Delta_0)}(\phi^q(\cdot - \varepsilon_i)) \quad \text{and} \quad \tilde{g}_i := P_{\tilde{V}_1(\Delta_0)}^{V_1(\Delta_0)}(\tilde{\phi}^q(\cdot - \varepsilon_i)) \tag{4.1}$$

for $i = 0, 1, 2$. Then define the subspaces X of $V_1(\Delta_0)$ and \tilde{X} of $\tilde{V}_1(\Delta_0)$ by

$$X := \text{span}(\{g_i : i = 0, 1, 2\} \cup \{\phi^i : i = 1, \dots, t\}) \quad \text{and} \\ \tilde{X} := \text{span}(\{\tilde{g}_i : i = 0, 1, 2\} \cup \{\tilde{\phi}^i : i = 1, \dots, t\}).$$

Let B be a basis for the space $(I - P_X^{\tilde{X}})V_1(\Delta_0)$ and let \tilde{B} be a basis for the space $(I - P_{\tilde{X}}^X)\tilde{V}_1(\Delta_0)$. Note that the elements of B are orthogonal to \tilde{V}_0 and the elements of \tilde{B} are orthogonal to V_0 by definition. Also notice that due to their support, the elements of both B and \tilde{B} are orthogonal to their own translates.

A small lemma is needed before we proceed.

LEMMA 4.1. $B \cap \tilde{B}^\perp = \{0\}$.

Proof. Let $\mathcal{W} := \{\phi^i : i = 1, \dots, t\}$ and $\tilde{\mathcal{W}} := \{\tilde{\phi}^i : i = 1, \dots, t\}$ and notice that, from the construction of the scaling functions, \mathcal{W} and $\tilde{\mathcal{W}}$ are biorthogonal sets. Consider $\psi \in B \cap \tilde{B}^\perp$. Then $\text{supp}(\psi) \subset \Delta_0$ and $\psi \in P_X^{\tilde{X}}V_1(\Delta_0)$. Then ψ is a linear combination of elements in \tilde{X} orthogonal to X .

Consider $\langle \psi, \phi^i \rangle$, $i = 1, \dots, t$. Since $\{\tilde{g}_i : i = 0, 1, 2\} \perp \mathcal{W}$, ψ is a linear combination of elements in $\tilde{\mathcal{W}}$. But, by Lemma 1.1, $\tilde{\mathcal{W}} \cap \mathcal{W}^\perp = 0$, so $\psi = 0$. ■

Then from Lemmas 4.1 and 1.1, there exist dual biorthogonal bases for B and \tilde{B} , denoted Ψ_f^A and $\tilde{\Psi}_f^A$, respectively. Recall the notation $\check{f} = f \circ r$, where r is the affine transformation from ∇_0 to Δ_0 for $f \in L^2(\nabla_0)$. Define $\Psi_f^\nabla := \{\check{\psi} : \psi \in \Psi_f^A\}$ and $\tilde{\Psi}_f^\nabla := \{\check{\tilde{\psi}} : \tilde{\psi} \in \tilde{\Psi}_f^A\}$. Define $W_f := S(\Psi_f^A \cup \Psi_f^\nabla) \subset \tilde{W}_0$ and $\tilde{W}_f := S(\tilde{\Psi}_f^A \cup \tilde{\Psi}_f^\nabla) \subset \tilde{W}_0$. The spaces W_f and \tilde{W}_f each have

$$2 \left(tN^2 + \frac{(N-1)(N-2)}{2} - (t+3) \right) = q(N^2 - 1) - 3N - 3$$

generators.

Before proceeding, the following lemmas are needed.

LEMMA 4.2. For g_i and \tilde{g}_i , $i = 0, 1, 2$, as defined in (4.1), $\langle g_i, \tilde{g}_j \rangle < 0$ for $i \neq j$.

Proof. Define $z_i := \phi^q(\cdot - \varepsilon_i)|_{\Delta_0}$ and $\tilde{z}_i := \tilde{\phi}^q(\cdot - \varepsilon_i)|_{\Delta_0}$ for $i = 0, 1, 2$. Recall that ϕ^q is the only scaling function with support larger than one $\Delta \in \mathcal{T}$. Notice that the z_i are still linear and nonnegative on all edges of Δ_0 .

Consider $\langle g_0, \tilde{g}_1 \rangle$. Express both z_0 and \tilde{z}_1 in terms of basis functions for $V_1|_{\Delta_0}$ and the \tilde{g}_i :

$$\begin{aligned} z_0 &= \phi^q \circ d_{0,0}|_{\Delta_0} + \frac{1}{N} \sum_{i=1}^{N-1} (N-i) \phi^q \circ d_{0,i}|_{\Delta_0} \\ &\quad + \frac{1}{N} \sum_{i=1}^{N-1} (N-i) \phi^q \circ d_{i,0}|_{\Delta_0} + g_0, \\ \tilde{z}_1 &= \tilde{\phi}^q \circ d_{N,0}|_{\Delta_0} + \frac{1}{N} \sum_{i=1}^{N-1} i \tilde{\phi}^q \circ d_{i,0}|_{\Delta_0} + \frac{1}{N} \sum_{i=1}^{N-1} i \tilde{\phi}^q \circ d_{i,N-i}|_{\Delta_0} + \tilde{g}_1. \end{aligned}$$

Recall that ϕ^q is orthogonal to the translates of $\tilde{\phi}^q$, even when restricted to bounded domains, so the same will be true for $\phi^q \circ d_{0,0}$ and $\tilde{\phi}^q \circ d_{0,0}$. Therefore, many of the terms of $\langle g_0, \tilde{g}_1 \rangle$ will vanish:

$$\langle z_0, \tilde{z}_1 \rangle = \frac{1}{N^2} \sum_{i=1}^{N-1} i(N-i) \langle \phi^q \circ d_{i,0}|_{\Delta_0}, \tilde{\phi}^q \circ d_{i,0}|_{\Delta_0} \rangle + \langle g_0, \tilde{g}_1 \rangle.$$

Since

$$\begin{aligned} \langle \phi^q \circ d_{i,0}|_{\Delta_0}, \tilde{\phi}^q \circ d_{i,0}|_{\Delta_0} \rangle &= K \langle \phi^q \circ d_{i,0}, \tilde{\phi}^q \circ d_{i,0} \rangle \\ &= \frac{K}{N^2} \langle \phi^q, \tilde{\phi}^q \rangle = \frac{K}{N^2} \end{aligned}$$

for some $0 < K < 1$, then

$$\langle z_0, \tilde{z}_1 \rangle = \frac{K}{N^4} \sum_{i=1}^{N-1} i(N-i) + \langle g_0, \tilde{g}_1 \rangle.$$

Using the identities

$$\sum_{i=1}^n i = \frac{n(n+1)}{2} \quad \text{and} \quad \sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6},$$

then

$$\langle z_0, \tilde{z}_1 \rangle = \frac{K(N^2 - 1)}{6N^3} + \langle g_0, \tilde{g}_1 \rangle. \tag{4.2}$$

Therefore, for $N > 1$,

$$\langle g_0, \tilde{g}_1 \rangle < 0.$$

It is easily verified that the same result holds for the remaining $\langle g_i, \tilde{g}_j \rangle, i \neq j$. ■

LEMMA 4.3. For g_i and $\tilde{g}_i, i = 0, 1, 2$, as defined in (4.1), the sets $\{g_0, g_1, g_2\}$ and $\{\tilde{g}_0, \tilde{g}_1, \tilde{g}_2\}$ are each linearly independent.

Proof. This proof hinges on the linear algebra result that for an n -dimensional space A and a space B where $A \cap B = \{0\}$, then $(I - P_B)A$ is an n -dimensional space. Recall the linear polynomials $h_i, i = 0, 1, 2$, supported on Δ_0 , and define the 3-dimensional space $H := \text{span}\{h_0, h_1, h_2\}$. Let $H^* := P_{V_1(\Delta_0)}H$. Since $H \cap (H - H^*) = \{0\}$, then $H^* = (I - (I - P_{V_1(\Delta_0)}))H$ is a 3-dimensional space.

Recall the space \mathcal{W} used in the construction of ϕ^q . Since $H^* \cap \mathcal{W} = \{0\}$, then

$$G := \text{span}\{g_0, g_1, g_2\} = (I - P_{\mathcal{W}})H^*$$

is a 3-dimensional space. An analogous proof holds for $\{\tilde{g}_0, \tilde{g}_1, \tilde{g}_2\}$. ■

LEMMA 4.4. For g_i and \tilde{g}_i , $i = 0, 1, 2$, as defined in (4.1), there exist σ_i and $\tilde{\sigma}_i$, $i = 0, 1, 2$, such that

1. $\text{span}\{\sigma_0, \sigma_1, \sigma_2\} = \text{span}\{g_0, g_1, g_2\}$,
2. $\text{span}\{\tilde{\sigma}_0, \tilde{\sigma}_1, \tilde{\sigma}_2\} = \text{span}\{\tilde{g}_0, \tilde{g}_1, \tilde{g}_2\}$,
3. $\{\sigma_0, \sigma_1, \sigma_2\}$ and $\{\tilde{\sigma}_0, \tilde{\sigma}_1, \tilde{\sigma}_2\}$ are biorthogonal sets, and
4. $\sigma_i \perp \tilde{g}_i$ and $\tilde{\sigma}_i \perp g_i$ for $i = 0, 1, 2$.

Proof. Note that, from Lemmas 4.2 and 4.3, the sets $G := \text{span}\{g_0, g_1, g_2\}$ and $\tilde{G} := \text{span}\{\tilde{g}_0, \tilde{g}_1, \tilde{g}_2\}$ are each 3-dimensional, but not biorthogonal. Define the following biorthogonal bases for G and \tilde{G} :

$$\begin{aligned} v_2 &:= g_0, & \tilde{v}_2 &:= \tilde{g}_0, \\ v_0 &:= g_1 - \frac{\langle g_1, \tilde{v}_2 \rangle}{\langle v_2, \tilde{v}_2 \rangle} v_2, & \tilde{v}_0 &:= \tilde{g}_1 - \frac{\langle \tilde{g}_1, v_2 \rangle}{\langle v_2, \tilde{v}_2 \rangle} \tilde{v}_2, \\ v_1 &:= g_2 - \frac{\langle g_2, \tilde{v}_2 \rangle}{\langle v_2, \tilde{v}_2 \rangle} v_2 - \frac{\langle g_2, \tilde{v}_0 \rangle}{\langle v_0, \tilde{v}_0 \rangle} v_0 & \text{and} & \tilde{v}_1 := \tilde{g}_2 - \frac{\langle \tilde{g}_2, v_2 \rangle}{\langle v_2, \tilde{v}_2 \rangle} \tilde{v}_2 - \frac{\langle \tilde{g}_2, v_0 \rangle}{\langle v_0, \tilde{v}_0 \rangle} \tilde{v}_0. \end{aligned}$$

Since each \tilde{v}_i is easily replaced with its additive inverse, assume without loss of generality that $\langle v_i, \tilde{v}_i \rangle > 0$.

Let $u_i := v_i$ and $\tilde{u}_i := \frac{\tilde{v}_i}{\langle v_i, \tilde{v}_i \rangle}$ for $i = 0, 1, 2$. Define the transformations $T : G \rightarrow \mathbf{R}^3$ and $\tilde{T} : \tilde{G} \rightarrow \mathbf{R}^3$ by

$$T(f) := (\langle f, \tilde{u}_0 \rangle, \langle f, \tilde{u}_1 \rangle, \langle f, \tilde{u}_2 \rangle)^T \text{ and } \tilde{T}(f) := (\langle f, u_0 \rangle, \langle f, u_1 \rangle, \langle f, u_2 \rangle)^T$$

so that

$$\begin{aligned} T(g_0) &= (0, 0, \langle g_0, \tilde{u}_2 \rangle)^T, & \tilde{T}(\tilde{g}_0) &= (0, 0, \langle \tilde{g}_0, u_2 \rangle)^T, \\ T(g_1) &= (\langle g_1, \tilde{u}_0 \rangle, 0, \langle g_1, \tilde{u}_2 \rangle)^T, & \tilde{T}(\tilde{g}_1) &= (\langle \tilde{g}_1, u_0 \rangle, 0, \langle \tilde{g}_1, u_2 \rangle)^T, \\ T(g_2) &= (\langle g_2, \tilde{u}_0 \rangle, \langle g_2, \tilde{u}_1 \rangle, \langle g_2, \tilde{u}_2 \rangle)^T \text{ and } & \tilde{T}(\tilde{g}_2) &= (\langle \tilde{g}_2, u_0 \rangle, \langle \tilde{g}_2, u_1 \rangle, \langle \tilde{g}_2, u_2 \rangle)^T. \end{aligned}$$

Define $\omega_0 \in G$ and $\tilde{\omega}_0 \in \tilde{G}$ by $T(\omega_0) := (\cos \theta, \sin \theta, 0)^T$ and $T(\tilde{\omega}_0) := (\cos \tilde{\theta}, \sin \tilde{\theta}, 0)^T$, respectively, so that $\omega_0 \perp \tilde{g}_0$ for all θ and $\tilde{\omega}_0 \perp g_0$ for all $\tilde{\theta}$. Then define $\omega_1 \in G$ and $\tilde{\omega}_1 \in \tilde{G}$ by

$$\begin{aligned} T(\omega_1) &:= T(\tilde{\omega}_0) \times T(\tilde{g}_1) \\ &= (\langle \tilde{g}_1, u_2 \rangle \sin \tilde{\theta}, -\langle \tilde{g}_1, u_2 \rangle \cos \tilde{\theta}, -\langle \tilde{g}_1, u_0 \rangle \sin \tilde{\theta})^T \quad \text{and} \\ T(\tilde{\omega}_1) &:= T(\omega_0) \times T(g_1) \\ &= (\langle g_1, \tilde{u}_2 \rangle \sin \theta, -\langle g_1, \tilde{u}_2 \rangle \cos \theta, -\langle g_1, \tilde{u}_0 \rangle \sin \theta)^T \end{aligned}$$

so that $\omega_1 \perp \tilde{\omega}_0$, $\omega_1 \perp \tilde{g}_1$, $\tilde{\omega}_1 \perp \omega_0$, and $\tilde{\omega}_1 \perp g_1$. Also, define $\omega_2 \in G$ and $\tilde{\omega}_2 \in \tilde{G}$ by

$$\begin{aligned} T(\omega_2) &:= T(\tilde{\omega}_0) \times T(\tilde{g}_2) \\ &= (\langle \tilde{g}_2, u_2 \rangle \sin \tilde{\theta}, -\langle \tilde{g}_2, u_2 \rangle \cos \tilde{\theta}, \langle \tilde{g}_2, u_1 \rangle \cos \tilde{\theta} - \langle \tilde{g}_2, u_0 \rangle \sin \tilde{\theta})^T \text{ and} \\ T(\tilde{\omega}_2) &:= T(\omega_0) \times T(g_2) \\ &= (\langle g_2, \tilde{u}_2 \rangle \sin \theta, -\langle g_2, \tilde{u}_2 \rangle \cos \theta, \langle g_2, \tilde{u}_1 \rangle \cos \theta - \langle g_2, \tilde{u}_0 \rangle \sin \theta)^T \end{aligned}$$

so that $\omega_2 \perp \tilde{\omega}_0$, $\omega_2 \perp \tilde{g}_2$, $\tilde{\omega}_2 \perp \omega_0$, and $\tilde{\omega}_2 \perp g_2$. Then $\omega_1 \perp \tilde{\omega}_2$ and $\tilde{\omega}_1 \perp \omega_2$ provided that there exist θ and $\tilde{\theta}$ such that $\langle T(\omega_1), T(\tilde{\omega}_2) \rangle = 0$ and $\langle T(\tilde{\omega}_1), T(\omega_2) \rangle = 0$; that is,

$$\begin{aligned} (\langle \tilde{g}_1, u_0 \rangle \langle g_2, \tilde{u}_0 \rangle + \langle \tilde{g}_1, u_2 \rangle \langle g_2, \tilde{u}_2 \rangle) \sin \theta \sin \tilde{\theta} + \langle \tilde{g}_1, u_2 \rangle \langle g_2, \tilde{u}_2 \rangle \cos \theta \cos \tilde{\theta} \\ - \langle \tilde{g}_1, u_0 \rangle \langle g_2, \tilde{u}_1 \rangle \sin \tilde{\theta} \cos \theta = 0, \end{aligned} \quad (4.3)$$

$$\begin{aligned} (\langle g_1, \tilde{u}_0 \rangle \langle \tilde{g}_2, u_0 \rangle + \langle g_1, \tilde{u}_2 \rangle \langle \tilde{g}_2, u_2 \rangle) \sin \theta \sin \tilde{\theta} + \langle g_1, \tilde{u}_2 \rangle \langle \tilde{g}_2, u_2 \rangle \cos \theta \cos \tilde{\theta} \\ - \langle g_1, \tilde{u}_0 \rangle \langle \tilde{g}_2, u_1 \rangle \sin \theta \cos \tilde{\theta} = 0, \end{aligned} \quad (4.4)$$

respectively.

Let $K_{i,j} := -\langle g_i, \tilde{g}_j \rangle > 0$ for $i \neq j$ from Lemma 4.2 and let $M_i := \langle g_i, \tilde{g}_i \rangle$ and $V_i := \langle v_i, \tilde{v}_i \rangle$ for $i = 0, 1, 2$. Note that $K_{0,1} = K_{1,2} = K_{2,0}$ and $K_{0,2} = K_{1,0} = K_{2,1}$ due to the rotational invariance of the inner product. Then

$$V_2 = M_0, \quad V_0 = \frac{M_0 M_1 - K_{0,1} K_{0,2}}{M_0}$$

and

$$V_1 = \frac{M_0 M_1 M_2 - K_{0,1} K_{0,2} (M_0 + M_1 + M_2) - K_{0,1}^3 - K_{0,2}^3}{M_0 M_1 - K_{0,1} K_{0,2}},$$

and so

$$\begin{aligned} \langle g_1, \tilde{u}_0 \rangle &= 1, & \langle \tilde{g}_1, u_0 \rangle &= V_0, \\ \langle g_1, \tilde{u}_2 \rangle &= -\frac{K_{0,2}}{V_2}, & \langle \tilde{g}_1, u_2 \rangle &= -K_{0,1}, \\ \langle g_2, \tilde{u}_0 \rangle &= -\frac{K_{0,1}^2 + V_2 K_{0,2}}{V_0 V_2}, & \langle \tilde{g}_2, u_0 \rangle &= -\frac{K_{0,2}^2 + V_2 K_{0,1}}{V_2}, \\ \langle g_2, \tilde{u}_1 \rangle &= 1, & \langle \tilde{g}_2, u_1 \rangle &= V_1, \\ \langle g_2, \tilde{u}_2 \rangle &= -\frac{K_{0,1}}{V_2} \quad \text{and} & \langle \tilde{g}_2, u_2 \rangle &= -K_{0,2}. \end{aligned}$$

Then Eqs. (4.3) and (4.4) reduce to

$$-K_{0,2}V_2 \sin \theta \sin \tilde{\theta} + K_{0,1}^2 \cos \theta \cos \tilde{\theta} + V_0V_2 \cos \theta \sin \tilde{\theta} = 0 \quad \text{and} \quad (4.5)$$

$$-K_{0,1}V_2 \sin \theta \sin \tilde{\theta} + K_{0,2}^2 \cos \theta \cos \tilde{\theta} + V_1V_2 \sin \tilde{\theta} \cos \theta = 0, \quad (4.6)$$

respectively.

Solving (4.5) and (4.6) for θ in terms of $\tilde{\theta}$ yields the conditions

$$\tan \theta = \frac{K_{0,1}^2 \cos \tilde{\theta} - V_0V_2 \sin \tilde{\theta}}{K_{0,2}V_2 \sin \tilde{\theta}} \quad \text{and} \quad \tan \theta = \frac{K_{0,2}^2 \cos \tilde{\theta}}{V_2K_{0,1} \sin \tilde{\theta} + V_1V_2 \cos \tilde{\theta}}, \quad (4.7)$$

respectively. Combining the two equations in (4.7) yields the single condition

$$K_{0,1}V_0V_2 \tan^2 \tilde{\theta} + (K_{0,2}^3 - K_{0,1}^3 + V_0V_1V_2) \tan \tilde{\theta} - K_{0,1}^2V_1 = 0,$$

a quadratic in $\tan \tilde{\theta}$ with real solutions.

Once θ and $\tilde{\theta}$ are found that satisfy (4.5) and (4.6), find the ω_i and $\tilde{\omega}_i$, $i = 0, 1, 2$, by the inverse transformations $T^{-1} : \mathbf{R}^3 \rightarrow G$ and $\tilde{T}^{-1} : \mathbf{R}^3 \rightarrow \tilde{G}$ defined by

$$T^{-1}(c_0, c_1, c_2) = c_0u_0 + c_1u_1 + c_2u_2 \quad \text{and}$$

$$\tilde{T}^{-1}(c_0, c_1, c_2) = c_0\tilde{u}_0 + c_1\tilde{u}_1 + c_2\tilde{u}_2.$$

As a final step, define $\sigma_i := \omega_i$ and $\tilde{\sigma}_i := \frac{\tilde{\omega}_i}{\langle \omega_i, \tilde{\omega}_i \rangle}$ for $i = 0, 1, 2$. ■

This lemma shows that each g_j is a linear combination of the two σ_j where $j \neq i$, and likewise for the \tilde{g}_i .

Consider the spaces

$$Y_0 := \text{span}\{\sigma_0, \check{\sigma}_1, \phi^q \circ d_{N-i,i} : i = 1, \dots, N-1\} \quad \text{and}$$

$$\tilde{Y}_0 := \text{span}\{\tilde{\sigma}_0, \check{\tilde{\sigma}}_1, \tilde{\phi}^q \circ d_{N-i,i} : i = 1, \dots, N-1\}.$$

Functions in Y_0 will be orthogonal to \tilde{W}_f and all translates of $\tilde{\phi}^i$, $i = 1, \dots, q-1$ and $\tilde{\phi}^q$ except $\tilde{\phi}^q \circ t_{1,0}$ and $\tilde{\phi}^q \circ t_{0,1}$. Likewise, functions in \tilde{Y}_0 will be orthogonal to W_f and all translates of ϕ^i , $i = 1, \dots, q-1$, and ϕ^q except $\phi^q \circ t_{1,0}$ and $\phi^q \circ t_{0,1}$. Define $X_0 := P_{\tilde{Y}_0}^{Y_0} V_0$ and $\tilde{X}_0 := P_{\tilde{Y}_0}^{Y_0} \tilde{V}_0$ to be two-dimensional subspaces of Y_0 and \tilde{Y}_0 , respectively. Let Ψ_0 and $\tilde{\Psi}_0$ be biorthogonal bases for the $(N-1)$ -dimensional complements $(I - P_{X_0}^{Y_0})Y_0$ and $(I - P_{\tilde{X}_0}^{\tilde{Y}_0})\tilde{Y}_0$. The elements of Ψ_0 satisfy all existing orthogonality conditions necessary to belong to the wavelet space W_0 , and likewise for $\tilde{\Psi}_0$.

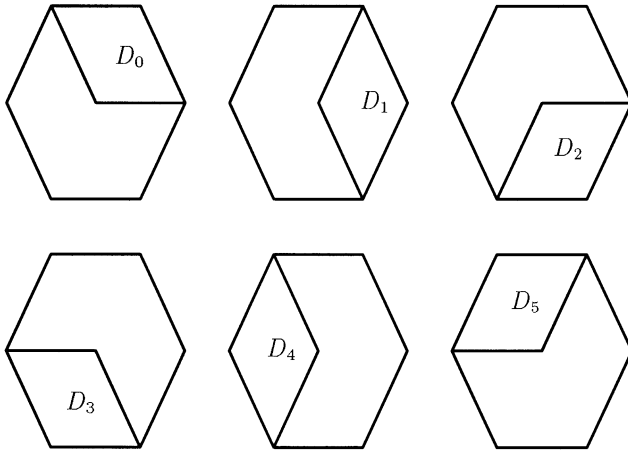


FIG. 5. Domains used in the construction of Ψ_h and $\tilde{\Psi}_h$.

The same construction can be used across the other two edges of Δ_0 using the spaces

$$\begin{aligned}
 Y_1 &:= \text{span}\{\sigma_2, \check{\sigma}_2 \circ t_{0,-1}, \phi^q \circ d_{i,0} : i = 1, \dots, N - 1\}, \\
 \tilde{Y}_1 &:= \text{span}\{\check{\sigma}_2, \check{\check{\sigma}}_2 \circ t_{0,-1}, \check{\phi}^q \circ d_{i,0} : i = 1, \dots, N - 1\}, \\
 Y_2 &:= \text{span}\{\sigma_1, \check{\sigma}_0 \circ t_{-1,0}, \phi^q \circ d_{0,i} : i = 1, \dots, N - 1\} \quad \text{and} \\
 \tilde{Y}_2 &:= \text{span}\{\check{\sigma}_1, \check{\check{\sigma}}_0 \circ t_{-1,0}, \check{\phi}^q \circ d_{0,i} : i = 1, \dots, N - 1\}
 \end{aligned}$$

and analogous subspaces $X_1, \tilde{X}_1, X_2,$ and \tilde{X}_2 to build biorthogonal pairs Ψ_1 and $\tilde{\Psi}_1$ and also Ψ_2 and $\tilde{\Psi}_2$. Define $\Psi_g := \Psi_0 \cup \Psi_1 \cup \Psi_2$ and $\tilde{\Psi}_g := \tilde{\Psi}_0 \cup \tilde{\Psi}_1 \cup \tilde{\Psi}_2$. The wavelets in Ψ_g and their translates will be orthogonal to the wavelets in $\tilde{\Psi}_g$ and their translates due to the biorthogonality of the σ_i and $\check{\sigma}_i$. Define $W_g := S(\Psi_g) \subset W_0$ and $\tilde{W}_g := S(\tilde{\Psi}_g) \subset \tilde{W}_0$. The spaces W_g and \tilde{W}_g each have $3(N - 1)$ generators.

Let D_0, \dots, D_5 be the parallelogram-shaped regions of \mathbf{R}^2 defined in Fig. 5. Define

$$v_i := P_{V_1(D_i)}^{\tilde{V}_1(D_i)} \phi^q \quad \text{and} \quad \tilde{v}_i := P_{\tilde{V}_1(D_i)}^{V_1(D_i)} \check{\phi}^q$$

for $i = 0, \dots, 5$ and consider for the moment v_0 . Notice that v_0 meets several orthogonality conditions required of wavelets in W_0 : $v_0 \perp \check{\phi}^j$ for $j = 1, \dots, \frac{q-1}{2}$, $v_0 \perp (\check{\phi}^j \circ t_{-1,0})$ for $j = \frac{q+1}{2}, \dots, q - 1$, $v_0 \perp (\check{\phi}^q \circ t_{0,1})$, and v_0 is perpendicular to \tilde{W}_g . Also, v_0 is perpendicular to generators of \tilde{W}_g that are built across the edges $(e_0, e_2), (e_1, e_2),$ and $(e_2, e_2 - e_1)$. Similar results can be found for the other v_i and \tilde{v}_i . The goal is to alter the v_i and \tilde{v}_i in such a way

that these orthogonalities are maintained, while achieving the other necessary orthogonalities.

Define

$$\mu_i := (I - P_{W_g(D_i)}^{W_g(D_i)})v_i + c_i\phi^q \circ d_{0,0} \quad \text{and} \quad \tilde{\mu}_i := (I - P_{\tilde{W}_g(D_i)}^{W_g(D_i)})\tilde{v}_i + \tilde{c}_i\tilde{\phi}^q \circ d_{0,0}$$

for $i = 0, \dots, 5$, where nonzero c_i and \tilde{c}_i satisfy $\langle \mu_i, \tilde{\phi}^q \rangle = 0$ and $\langle \tilde{\mu}_i, \phi^q \rangle = 0$, respectively. From Lemma 4.4, there exist biorthogonal sets $\Sigma = \{\sigma_0, \sigma_1, \sigma_2\}$ and $\tilde{\Sigma} = \{\tilde{\sigma}_0, \tilde{\sigma}_1, \tilde{\sigma}_2\}$ such that $\text{span}(\Sigma) = \text{span}\{g_0, g_1, g_2\}$, $\text{span}(\tilde{\Sigma}) = \text{span}\{\tilde{g}_0, \tilde{g}_1, \tilde{g}_2\}$, and $\sigma_j \perp \tilde{g}_j$ and $\tilde{\sigma}_j \perp g_j$ for $j = 0, 1, 2$. Then

$$\begin{aligned} \mu_0 &:= v_0 - \langle v_0, \tilde{\sigma}_2 \rangle \sigma_2 - \langle v_0, \tilde{\sigma}_1 \circ t_{-1,0} \rangle \tilde{\sigma}_1 \circ t_{-1,0} + c_0\phi^q \circ d_{0,0} \\ &= c_0\phi^q \circ d_{0,0} + \langle g_0, \tilde{\sigma}_1 \rangle \sigma_1 + \langle g_2, \tilde{\sigma}_0 \rangle \tilde{\sigma}_0 + \frac{1}{N} \sum_{j=1}^{N-1} (N-j)\phi^q \circ d_{0,j} \quad \text{and} \end{aligned} \tag{4.8}$$

$$\begin{aligned} \tilde{\mu}_1 &:= \tilde{v}_1 - \langle \tilde{v}_1, \sigma_1 \rangle \tilde{\sigma}_1 - \langle \tilde{v}_1, \sigma_1 \circ t_{0,-1} \rangle \tilde{\sigma}_1 \circ t_{0,-1} + \tilde{c}_1\tilde{\phi}^q \circ d_{0,0} \\ &= \tilde{c}_1\tilde{\phi}^q \circ d_{0,0} + \langle \tilde{g}_0, \sigma_2 \rangle \tilde{\sigma}_2 + \langle \tilde{g}_0, \sigma_2 \rangle \tilde{\sigma}_2 \circ t_{0,-1} + \frac{1}{N} \sum_{j=1}^{N-1} (N-j)\tilde{\phi}^q \circ d_{j,0}. \end{aligned} \tag{4.9}$$

Notice that the μ_i and $\tilde{\mu}_i$ maintain the orthogonalities of the v_i and \tilde{v}_i . Also, by definition, $\mu_i \perp \tilde{\phi}^q$, $\tilde{\mu}_i \perp \phi^q$, $\mu_i \perp \tilde{W}_g$, and $\tilde{\mu}_i \perp W_g$. Finally, note from (4.8) and (4.9)

$$\begin{aligned} \langle \mu_0 \circ t_{1,-1}, \tilde{\mu}_1 \rangle &= \langle g_2, \tilde{\sigma}_0 \rangle \langle \tilde{g}_0, \sigma_2 \rangle \langle \tilde{\sigma}_0, \tilde{\sigma}_2 \rangle = 0 \quad \text{and} \\ \langle \mu_0, \tilde{\mu}_1 \rangle &= c_0\tilde{c}_1 + \langle g_0, \tilde{\sigma}_1 \rangle \langle \tilde{g}_0, \sigma_2 \rangle \langle \sigma_1, \tilde{\sigma}_2 \rangle = c_0\tilde{c}_1 \neq 0. \end{aligned}$$

Also, it is trivially established that $(\mu_0 \circ t_{1,0}) \perp \tilde{\mu}_1$ and $(\mu_0 \circ t_{0,1}) \perp \tilde{\mu}_1$. Similarly, it is established that the μ_i and $\tilde{\mu}_i$ satisfy the condition $\mu_i \perp (\tilde{\mu}_j \circ t_{m,n})$ for $m, n \neq 0$, $i, j = 0, \dots, 5$, $i \neq j$, and that the sets $\{\mu_i\}$ and $\{\tilde{\mu}_i\}$ satisfy Lemma 1.1. Let Ψ_h and $\tilde{\Psi}_h$ be biorthogonal bases for $\text{span}\{\mu_i : i = 0, \dots, 5\}$ and $\text{span}\{\tilde{\mu}_i : i = 0, \dots, 5\}$, respectively. Define $W_h := S(\Psi_h) \subset W_0$ and $\tilde{W}_h := S(\tilde{\Psi}_h) \subset \tilde{W}_0$.

Before establishing that all the wavelets necessary to “build” V_1 and \tilde{V}_1 have been found, a lemma is needed.

LEMMA 4.5. *For g_i and \tilde{g}_i , $i = 0, 1, 2$, as defined in (4.1), $g_i \in V_0 + W_g + W_h$ and $\tilde{g}_i \in \tilde{V}_0 + \tilde{W}_g + \tilde{W}_h$ for $i = 0, 1, 2$.*

Proof. By Lemma 4.4, it suffices to show that $\sigma_i \in V_0 + W_g + W_h$ and $\tilde{\sigma}_i \in \tilde{V}_0 + \tilde{W}_g + \tilde{W}_h$ for $i = 0, 1, 2$. Recall the space Y_0 is spanned by σ_1 , $\tilde{\sigma}_0 \circ t_{-1,0}$, and the functions $\phi^q \circ d_{0,i}$, $i = 1, \dots, N-1$, and, likewise, \tilde{Y}_0 is spanned by $\tilde{\sigma}_1$, $\tilde{\sigma}_0 \circ t_{-1,0}$, and the functions $\tilde{\phi}^q \circ d_{0,i}$, $i = 1, \dots, N-1$.

Recall that X_0 is the span of $P_{Y_0}^{\tilde{Y}_0} \phi^q$ and $P_{Y_0}^{\tilde{Y}_0}(\phi^q \circ t_{0,1})$ and likewise, \tilde{X}_0 is the span of $P_{\tilde{Y}_0}^{Y_0} \tilde{\phi}^q$ and $P_{\tilde{Y}_0}^{Y_0}(\tilde{\phi}^q \circ t_{0,1})$. By definition, $\sigma_1 \in Y_0$ and $(I - P_{X_0}^{\tilde{Y}_0})Y_0 \subset W_g$. It suffices to show that $X_0 \subset V_0 + W_h$.

It is tediously verified that

$$\begin{aligned}
 & c_0(\phi^q - \mu_1 - \mu_2 - \mu_3 - \mu_4 - \mu_5) - (1 - c_1 - c_2 - c_3 - c_4 - c_5)\mu_0 \\
 &= c_0 \langle g_0, \tilde{\sigma}_1 \rangle \sigma_1 + c_0 \langle g_2, \tilde{\sigma}_0 \rangle \check{\sigma}_0 \circ t_{-1,0} + \frac{c_0}{N} \sum_{i=1}^{N-1} (N-i)\phi^q \circ d_{0,i} \\
 & \quad + (c_1 + c_2 + c_3 + c_4 + c_5 - 1) \langle g_0, \tilde{\sigma}_1 \rangle \sigma_1 \\
 & \quad + (c_1 + c_2 + c_3 + c_4 + c_5 - 1) \langle g_2, \tilde{\sigma}_0 \rangle \check{\sigma}_0 \circ t_{-1,0} \\
 & \quad + \left(\frac{c_1 + c_2 + c_3 + c_4 + c_5 - 1}{N} \right) \sum_{i=1}^{N-1} (N-i)\phi^q \circ d_{0,i} \\
 &= (c_0 + c_1 + c_2 + c_3 + c_4 + c_5 - 1) [\langle g_0, \tilde{\sigma}_1 \rangle \sigma_1 \\
 & \quad + \langle g_2, \tilde{\sigma}_0 \rangle \check{\sigma}_0 \circ t_{-1,0} + \frac{1}{N} \sum_{i=1}^{N-1} (N-i)\phi^q \circ d_{0,i}] \\
 &= (c_0 + c_1 + c_2 + c_3 + c_4 + c_5 - 1) P_{Y_0}^{\tilde{Y}_0} \phi^q.
 \end{aligned}$$

Likewise,

$$\begin{aligned}
 & c_3(\phi^q - \mu_0 - \mu_1 - \mu_2 - \mu_4 - \mu_5) - (1 - c_0 - c_1 - c_2 - c_4 - c_5)\mu_3 \circ t_{0,1} \\
 &= (c_0 + c_1 + c_2 + c_3 + c_4 + c_5 - 1) P_{Y_0}^{\tilde{Y}_0}(\phi^q \circ t_{0,1}).
 \end{aligned}$$

Thus, $X_0 \subset V_0 + W_h$ and $\sigma_1 \in V_0 + W_g + W_h$. Analogous arguments establish $\sigma_0, \sigma_2 \in V_0 + W_g + W_h$ and $\tilde{\sigma}_i \in \tilde{V}_0 + \tilde{W}_g + \tilde{W}_h$. ■

THEOREM 4.6. *Let (V_p) and (\tilde{V}_p) be biorthogonal MRA of multiplicity r in \mathbf{R}^2 constructed from Theorem 2.1. Define $W_f, \tilde{W}_f, W_g, \tilde{W}_g, W_h,$ and \tilde{W}_h as above. Then $V_1 = V_0 + W_0$ and $\tilde{V}_1 = \tilde{V}_0 + \tilde{W}_0$ where $W_0 = W_f + W_g + W_h,$ $\tilde{W}_0 = \tilde{W}_f + \tilde{W}_g + \tilde{W}_h,$ and W_0 and \tilde{W}_0 each have $q(N^2 - 1)$ generators.*

Proof. Define $W := W_f + W_g + W_h, \tilde{W} := \tilde{W}_f + \tilde{W}_g + \tilde{W}_h, V := V_1(\Delta_0),$ and $\tilde{V} := \tilde{V}_1(\Delta_0)$. Certainly, $V_1 \supseteq V_0 + W$ by nature of the wavelet constructions. At issue is whether $V_1 \subseteq V_0 + W$.

For $N > 2$, generators $\phi^i \circ d_{0,0}, i = 1, \dots, q-1$ of V_1 can be found in the space V . Notice that

$$\dim V = tN^2 + \frac{(N-1)(N-2)}{2},$$

where $t = \frac{q-1}{2}$. The scaling functions and the definitions of W_f , \tilde{W}_f , W_g and \tilde{W}_g , along with Lemma 4.5, provide biorthogonal bases

$$\{\phi^1, \dots, \phi^t\} \cup \Psi_f \cup \{\sigma_0, \sigma_1, \sigma_2\} \quad \text{and} \quad \{\tilde{\phi}^1, \dots, \tilde{\phi}^t\} \cup \tilde{\Psi}_f \cup \{\tilde{\sigma}_0, \tilde{\sigma}_1, \tilde{\sigma}_2\}$$

of V and \tilde{V} , each with cardinality

$$t + \left(tN^2 + \frac{(N-1)(N-2)}{2} - t - 3 \right) + 3 = tN^2 + \frac{(N-1)(N-2)}{2}.$$

Since the linear systems have full rank, each $f \in V$ is a linear combination of elements of $V_0 + W$ and each $\tilde{f} \in \tilde{V}$ is a linear combination of elements of $\tilde{V}_0 + \tilde{W}$. Thus, $\phi^i \circ d_{0,0} \in V_0 + W$ and $\tilde{\phi}^i \circ d_{0,0} \in \tilde{V}_0 + \tilde{W}$ for $i = 1, \dots, q-1$.

Also notice that

$$\phi^q - \sum_{i=0}^5 \mu_i = \left(1 - \sum_{i=0}^5 c_i \right) \phi^q \circ d_{0,0}.$$

Thus, $V_1 \subseteq V_0 + W_f + W_g + W_h$ and $W = W_0$. The analogous results hold for \tilde{W} . The number of generators is the sum of the generators for W_f , W_g , and W_h :

$$(q(N^2 - 1) - 3N - 3) + (3(N - 1)) + 6 = q(N^2 - 1). \quad \blacksquare$$

COROLLARY 4.7. *Let (V_p) and (\tilde{V}_p) be biorthogonal MRA of multiplicity q in \mathbf{R}^2 constructed from Theorem 2.1. Define W_f , \tilde{W}_f , W_g , and \tilde{W}_g as above. Let D be the hexagonal support of ϕ^q , and let $X := (V_0 + W_f + W_g)(D)$ and $\tilde{X} := (\tilde{V}_0 + \tilde{W}_f + \tilde{W}_g)(D)$. Then W_h and \tilde{W}_h are generated by biorthogonal bases for $P_X^X V_1(D)$ and $P_{\tilde{X}}^{\tilde{X}} \tilde{V}_1(D)$.*

While the definitions of W_h and \tilde{W}_h provide an explicit construction, Corollary 4.7 says that after finding the generators of W_f , \tilde{W}_f , W_g and \tilde{W}_g , the generators of W_h are whatever is left in V_1 with the support of ϕ^q , and likewise for \tilde{W}_h .

5. WAVELETS FOR SCALING FUNCTIONS IN SECTION 3

By Theorem 4.6, W_0 and \tilde{W}_0 will each have $3(3^2 - 1) = 24$ generators. Let D_0 be the overlap domain with vertices at $(0, 0)$, $\varepsilon_1, \varepsilon_2$, and $\varepsilon_2 - \varepsilon_1$ and let D_i be clockwise rotation of D_0 about the origin by $\frac{\pi}{3}i$ for $i = 1, \dots, 5$, as in Fig. 5.

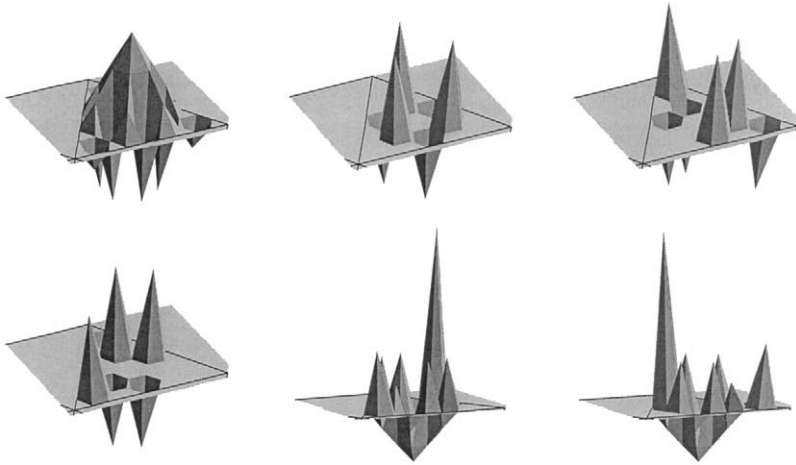


FIG. 6. Wavelets ψ^1 through ψ^6 .

5.1. Wavelets in W_f and \tilde{W}_f

By definition, W_f and \tilde{W}_f each have 12 generators, 6 supported on Δ_0 and 6 supported on ∇_0 . Define the 10-dimensional spaces $V := V_1(\Delta_0)$ and $\tilde{V} := \tilde{V}_1(\Delta_0)$, and the 4-dimensional spaces

$$\begin{aligned}
 X &:= P_V^{\tilde{V}}(\text{span}\{\phi^1, \phi^3, \phi^3 \circ t_{1,0}, \phi^3 \circ t_{0,1}\}) \quad \text{and} \\
 \tilde{X} &:= P_{\tilde{V}}^V(\text{span}\{\tilde{\phi}^1, \tilde{\phi}^3, \tilde{\phi}^3 \circ t_{1,0}, \tilde{\phi}^3 \circ t_{0,1}\}).
 \end{aligned}$$

The ψ^i , $i = 1, \dots, 6$, illustrated in Fig. 6, were chosen as a spanning set for $(I - P_X^{\tilde{X}})V$ that met some symmetry conditions. The associated $\tilde{\psi}^i$, illustrated in Fig. 7, were chosen so that $\tilde{\psi}^i \in \ker(\text{span}\{\psi^j : j \neq i\} \cup X)$ and $\langle \psi^i, \tilde{\psi}^i \rangle > 0$ for $i = 1, \dots, 6$. Each of the above is “normalized” by the factor $\sqrt{\langle \psi^i, \tilde{\psi}^i \rangle}$.

These wavelets reflected onto ∇_0 will span $W_f|_{\chi_{\nabla_0}}$ and $\tilde{W}_f|_{\chi_{\nabla_0}}$. Define

$$\psi^{i+6} := \psi^i \circ r \circ t_{-1,0} \quad \text{and} \quad \tilde{\psi}^{i+6} := \tilde{\psi}^i \circ r \circ t_{-1,0}$$

for $i = 1, \dots, 6$.

5.2. Wavelets in W_g and \tilde{W}_g

By definition, W_g and \tilde{W}_g each have 6 generators. Following the construction in the proof of Lemma 4.4, biorthogonal sets $\{\sigma_0, \sigma_1, \sigma_2\}$ and $\{\tilde{\sigma}_0, \tilde{\sigma}_1, \tilde{\sigma}_2\}$ can be found such that $\sigma_i \perp \tilde{g}_i$ and $\tilde{\sigma}_i \perp g_i$, $i = 0, 1, 2$. Then, following the construction in Section 3, functions in W_g with support on the parallelogram $(\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_2 - \varepsilon_1)$ will be linear combinations of

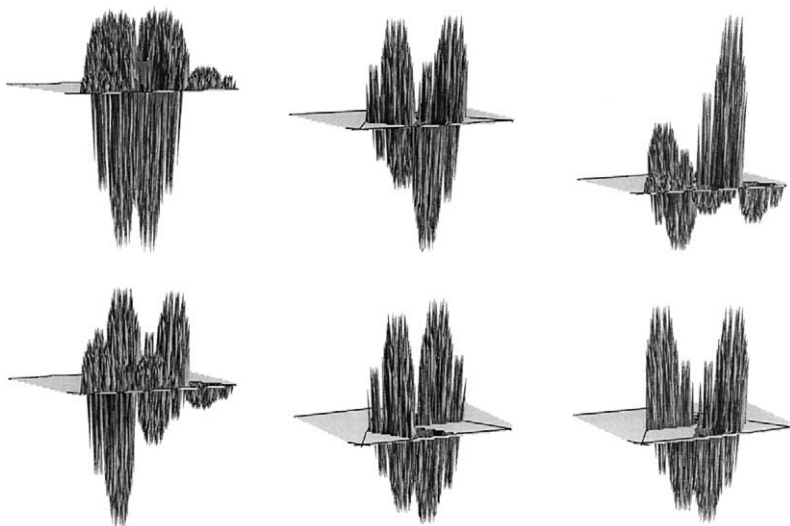


FIG. 7. Wavelets $\tilde{\psi}^1$ through $\tilde{\psi}^6$.

σ_1 , $\check{\sigma}_0 \circ t_{-1,0}$, $\phi^3 \circ d_{0,1}$, and $\phi^3 \circ d_{0,2}$. These functions are orthogonal to all translates of ϕ^1 , ϕ^2 , and ψ^i for $i = 1, \dots, 12$. Also, $\sigma_2 \perp \phi^3 \circ t_{1,0}$ and $\check{\sigma}_0 \circ t_{-1,0} \perp \phi^3 \circ t_{-1,1}$. Only four other orthogonality conditions must be met.

It is possible to construct symmetric–antisymmetric pairs of wavelets. Define

$$v_1 := \sigma_1 + \check{\sigma}_0 \circ t_{-1,0} + c_1 \phi^3 \circ d_{0,1} + c_2 \phi^3 \circ d_{0,2},$$

and solve the system of equations

$$\begin{cases} \langle v_1, \tilde{\phi}^3 \rangle = 0, \\ \langle v_1, \tilde{\phi}^3 \circ t_{0,1} \rangle = 0 \end{cases}$$

for c_1 and c_2 . Likewise, define

$$\tilde{v}_1 := \tilde{\sigma}_1 + \check{\check{\sigma}}_0 \circ t_{-1,0} + \tilde{c}_1 \tilde{\phi}^3 \circ d_{0,1} + \tilde{c}_2 \tilde{\phi}^3 \circ d_{0,2},$$

and solve the system of equations

$$\begin{cases} \langle \tilde{v}_1, \phi^3 \rangle = 0, \\ \langle \tilde{v}_1, \phi^3 \circ t_{0,1} \rangle = 0 \end{cases}$$

for \tilde{c}_1 and \tilde{c}_2 . If $\langle v_1, \tilde{v}_1 \rangle < 0$, then change \tilde{v}_1 to its additive inverse. Also, define v_2 and \tilde{v}_2 by

$$v_2 := \sigma_1 - \check{\sigma}_0 \circ t_{-1,0} \quad \text{and} \quad \tilde{v}_2 := \tilde{\sigma}_1 - \check{\check{\sigma}}_0 \circ t_{-1,0}.$$

Then $v_1 \perp \tilde{v}_2$ and $\tilde{v}_1 \perp v_2$ by nature of their symmetry properties.

The remaining wavelets generating W_g are merely $2\pi/3$ rotations of v_1 and v_2 about ε_0 , denoted as $\bar{\tau}$. Define

$$\begin{aligned} \omega^{13} &:= v_1, & \tilde{\omega}^{13} &:= \tilde{v}_1, \\ \omega^{14} &:= v_2, & \tilde{\omega}^{14} &:= \tilde{v}_2, \\ \omega^{15} &:= v_1 \circ \bar{\tau}, & \tilde{\omega}^{15} &:= \tilde{v}_1 \circ \bar{\tau}, \\ \omega^{16} &:= v_2 \circ \bar{\tau}, & \tilde{\omega}^{16} &:= \tilde{v}_2 \circ \bar{\tau}, \\ \omega^{17} &:= v_1 \circ \bar{\tau} \circ \bar{\tau}, & \tilde{\omega}^{17} &:= \tilde{v}_1 \circ \bar{\tau} \circ \bar{\tau}, \\ \omega^{18} &:= v_2 \circ \bar{\tau} \circ \bar{\tau} \quad \text{and} \quad \tilde{\omega}^{18} &:= \tilde{v}_2 \circ \bar{\tau} \circ \bar{\tau}. \end{aligned}$$

Normalize by defining $\psi^i := \omega^i / \sqrt{\langle \omega^i, \tilde{\omega}^i \rangle}$ and $\tilde{\psi}^i := \tilde{\omega}^i / \sqrt{\langle \omega^i, \tilde{\omega}^i \rangle}$ for $i = 13, \dots, 18$. Wavelets $\psi^{13}, \psi^{14}, \tilde{\psi}^{13}$, and $\tilde{\psi}^{14}$ are illustrated in Fig. 8.

5.3. Wavelets in W_h and \tilde{W}_h

By definition, W_h has 6 generators. Following the construction in Section 3, construct μ_i and $\tilde{\mu}_i$, $i = 0, \dots, 5$ that span W_h and \tilde{W}_h , respectively. It can be verified that $c_0 = c_2 = c_4$, $c_1 = c_3 = c_5$, $\check{c}_0 = \check{c}_2 = \check{c}_4$, and $\check{c}_1 = \check{c}_3 = \check{c}_5$ due to the rotational invariance of both the g_i and \check{g}_i . To construct biorthogonal sets with some symmetric properties, first define the following:

$$\begin{aligned} \gamma_1 &:= \sum_{i=0}^5 \mu_i, & \tilde{\gamma}_1 &:= \sum_{i=0}^5 \tilde{\mu}_i, \\ \gamma_2 &:= \sum_{i=0}^5 (-1)^i \mu_i, & \tilde{\gamma}_2 &:= \sum_{i=0}^5 (-1)^i \tilde{\mu}_i, \\ \gamma_3 &:= \mu_0 - \mu_2, & \tilde{\gamma}_3 &:= \tilde{\mu}_0 - \tilde{\mu}_2. \\ \gamma_4 &:= \mu_1 - \mu_3, & \tilde{\gamma}_4 &:= \tilde{\mu}_1 - \tilde{\mu}_3, \\ \gamma_5 &:= \mu_0 + \mu_2, & \tilde{\gamma}_5 &:= \tilde{\mu}_0 + \tilde{\mu}_2, \\ \gamma_6 &:= \mu_1 + \mu_3, & \tilde{\gamma}_6 &:= \tilde{\mu}_1 + \tilde{\mu}_3 \end{aligned}$$

Then construct the biorthogonal sets $\{\omega_1, \dots, \omega_6\}$ and $\{\tilde{\omega}_1, \dots, \tilde{\omega}_6\}$ using the biorthogonal Gram–Schmidt process so that $\langle \omega_i, \omega_i \rangle > 0$, $i = 1, \dots, 6$.

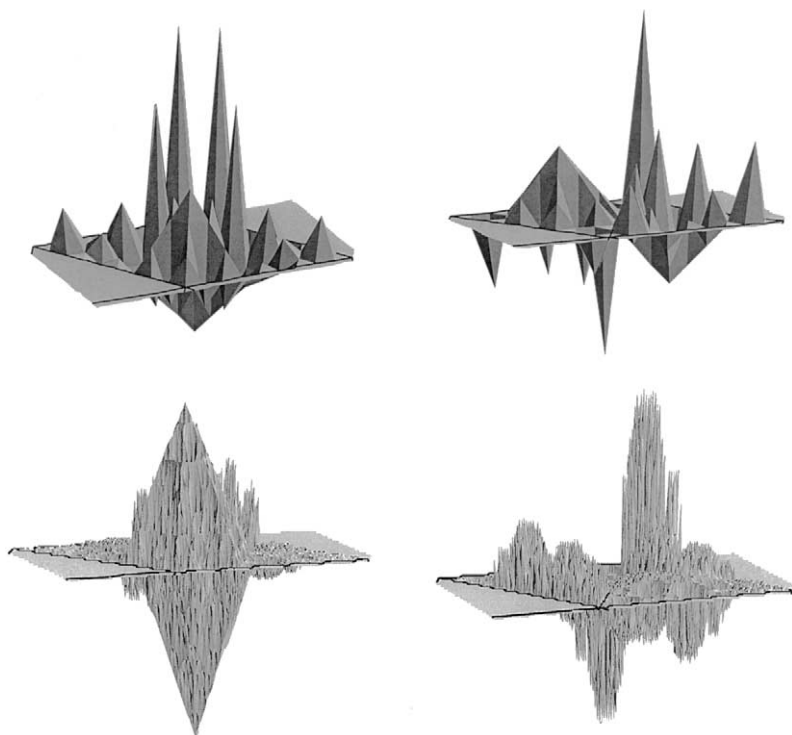


FIG. 8. Wavelets ψ^{13} , ψ^{14} , $\tilde{\psi}^{13}$, and ψ^{14} .

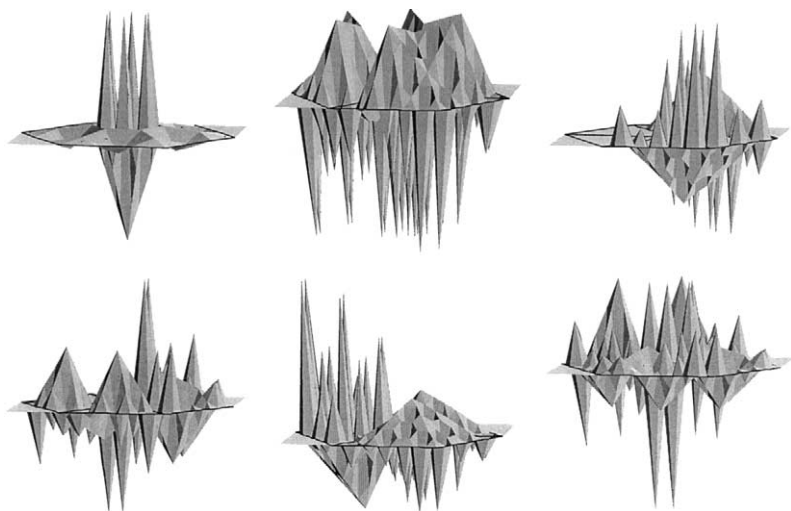


FIG. 9. Wavelets ψ^{19} through ψ^{24} .

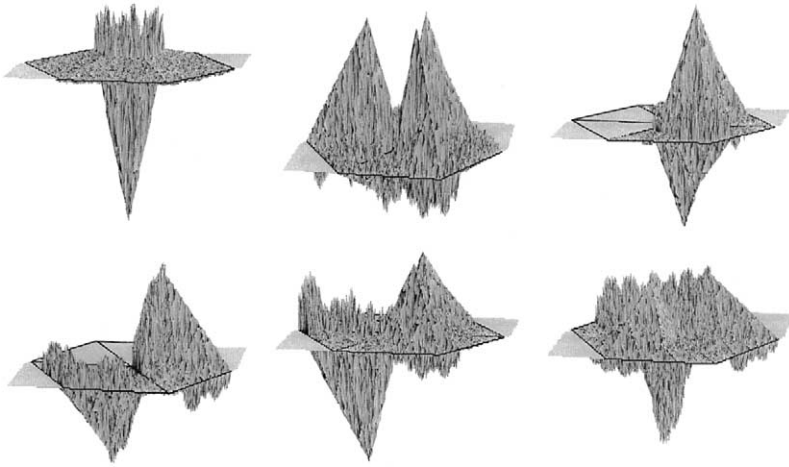


FIG. 10. Wavelets $\tilde{\psi}^{19}$ through $\tilde{\psi}^{24}$.

Define $\psi^{i+18} := \frac{\omega_i}{\sqrt{\langle \omega_i, \omega_i \rangle}}$ and $\tilde{\psi}^{i+18} := \frac{\tilde{\omega}_i}{\sqrt{\langle \tilde{\omega}_i, \tilde{\omega}_i \rangle}}$ for $i = 1, \dots, 6$. These wavelets are illustrated in Figs. 9 and 10, respectively. The sets $S(\{\psi^i : i = 1, \dots, 24\})$ and $S(\{\tilde{\psi}^i : i = 1, \dots, 24\})$ form biorthogonal bases for W_0 and \tilde{W}_0 .

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